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Energy and Momentum Densities of Cosmological Models, with Equation of State $\rho = \mu$, in General Relativity and Teleparallel Gravity

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Abstract We calculated the energy and momentum densities of stiff fluid solutions, using Einstein, Bergmann–Thomson and Landau–Lifshitz energy-momentum complexes, in both general relativity and teleparallel gravity. In our analysis we get different results comparing the aforementioned complexes with each other when calculated in the same gravitational theory, either this is in general relativity and teleparallel gravity. However, interestingly enough, each complex's value is the same either in general relativity or teleparallel gravity. Our results sustain that (i) general relativity or teleparallel gravity are equivalent theories (ii) different energy-momentum complexes do not provide the same energy and momentum densities neither in general relativity nor in teleparallel gravity. In the context of the theory of teleparallel gravity, the vector and axial-vector parts of the torsion are obtained. We show that the axial-vector torsion vanishes for the space-time under study.

Keywords Energy and momentum densities · Stiff fluid solutions · Teleparallel gravity

1 Introduction

The issue of energy localization was first discussed during the early years after the development of general relativity and debate continued for decades. There are different attempts to find a general accepted definition of the energy density for the gravitational field. However, there is still no generally accepted definition known. The foremost endeavor was made by Einstein [1] who suggested a definition for energy-momentum distribution. Following this definition, many physicists proposed different energy-momentum complexes: e.g. Tolman [2], Landau and Lifshitz [3], Papapetrou [4], Bergmann and Thomson [5], Weinberg [6] and Møller [7]. Except for the Møller definition, others are restricted to calculate the energy and momentum distributions in quasi-Cartesian coordinates to get a reasonable and meaningful result.

Despite these drawbacks, some interesting results obtained recently leads to the conclusion that these definitions give exactly the same energy distribution for any given space-time

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[8–26]. However, some examples of space-times have been explored which do not support these results [27–31].

The problem of energy-momentum localization can also be reformulated in the context of teleparallel gravity [32]. By working in the context of teleparallel gravity, Vargas [32] obtained the teleparallel version of both Einstein and Landau–Lifshitz energy-momentum complexes. He used these definitions and found that the total energy is zero in Friedmann–Robertson–Walker space-time. His results are the same as those calculated in general relativity. Salti and his collaborators [33–36] considered different space-times for various definitions in teleparallel gravity to obtain the energy-momentum distribution in a given model. Their results agree with the previous results obtained in the theory of general relativity.

The paper is organized as follows: In the next section we briefly present the cosmological model, whose source is a stiff fluid, in which the energy and momentum densities by using different prescriptions are to be calculated. In three subsections of Sect. 3, we explicitly compute the energy and momentum densities of the space-time under consideration, using Einstein, Bergmann–Thomson and Landau–Lifshitz energy momentum complexes, in the context of general relativity. In Sect. 4, we briefly present the concept of energy-momentum complexes in the context of teleparallel gravity theory. In the subsequent section, we calculate the energy and momentum densities in teleparallel gravity, using the aforementioned complexes. In Sect. 6, we obtain the vector and axial-vector parts of the torsion in the theory of teleparallel gravity. Finally, in Sect. 7, a brief summary of results and concluding remarks are presented.

Through this paper we will use $G = 1$ and $c = 1$ units and the Greek alphabet ($\mu, \nu, \rho = 0, 1, 2, 3$) to denote tensor indices, that is, indices related to space-time. The Latin alphabet ($a, b, c, = 0, 1, 2, 3$) will be used to denote local Lorentz (or tangent space) indices, whose associated metric tensor is $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$.

2 Stiff Fluid Solutions

The cylindrically symmetric solutions of the Einstein equations whose source is a "stiff fluid", that is, a perfect fluid with the equation of state, $\rho = p$, can be written in the form

$$ds^2 = g_{00}(-dt^2 + dx^2) + g_{AB}dx^A dx^B, \tag{2.1}$$

where $A, B = 2, 3$, $(x^2, x^3) = (y, z)$ and all the components $g_{\alpha\beta}$ depend only on t and x.

The interest in vacuum solutions with the above metric stems from the fact that there exists an elaborate theory which tells how to generate new vacuum solutions of the form (2.1) from known solutions by algebraic operations (the technique and its applications were given by Verdaguer [37]).

The possibility of generating cylindrically symmetric "stiff fluid" solutions from vacuum solutions was apparently first indicated by Wainwright et al. [38]. These solutions have all been generated by an algorithm.

The metric discussed by Wainwright et al. [38] is of the form

$$ds^2 = e^{2k+\Omega}(-dt^2 + dx^2) + R[f(dy + \omega dz)^2 + f^{-1}dz^2], \tag{2.2}$$

where k, Ω, R, f and ω are functions of t and x.

This metric admits the two commuting spacelike Killing vector fields[1]

$$\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z}.$$

Thus it must satisfy the field equation

$$\mathbf{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathbf{R} = 8\pi[(\rho + p)u_\mu u_\nu + p g_{\mu\nu}].$$

With

$$\rho = p.$$

The covariant and contravariant components of the metric (2.2), respectively, are given as follows

$$g_{\mu\nu} = -e^{2k+\Omega}\delta_\mu^0\delta_\nu^0 + e^{2k+\Omega}\delta_\mu^1\delta_\nu^1 + Rf\delta_\mu^2\delta_\nu^2 + (Rf\omega^2 + f^{-1})\delta_\mu^3\delta_\nu^3 + Rf\omega(\delta_\mu^2\delta_\nu^3 + \delta_\mu^3\delta_\nu^2), \tag{2.3}$$

$$g^{\mu\nu} = -e^{-(2k+\Omega)}\delta_0^\mu\delta_0^\nu + e^{-(2k+\Omega)}\delta_1^\mu\delta_1^\nu + \frac{f^2\omega^2 + 1}{Rf}\delta_2^\mu\delta_2^\nu + \frac{f}{R}\delta_3^\mu\delta_3^\nu - \frac{f\omega}{R}(\delta_2^\mu\delta_3^\nu + \delta_3^\mu\delta_2^\nu). \tag{2.4}$$

3 Energy-Momentum Complexes in General Relativity

3.1 Einstein's Energy-Momentum Complex

The energy-momentum complex as defined by Einstein [1] is given by

$$\theta_\mu^\nu = \frac{1}{16\pi} H_{\mu,\alpha}^{\nu\alpha}, \tag{3.1}$$

where the Einstein's superpotential $H_\mu^{\nu\alpha}$ is of the form

$$H_\mu^{\nu\alpha} = -H_\mu^{\alpha\nu} = \frac{g_{\mu\beta}}{\sqrt{-g}}\left[-g\left(g^{\nu\beta}g^{\alpha\rho} - g^{\alpha\beta}g^{\nu\rho}\right)\right]_{,\rho}. \tag{3.2}$$

θ_0^0 and θ_i^0 $(i = 1, 2, 3)$, are the energy and momentum density components, respectively. The energy-momentum complex θ_μ^ν satisfies the local conservation law

$$\frac{\partial\theta_\mu^\nu}{\partial x^\nu} = 0.$$

In order to evaluate the energy and momentum densities in Einstein's prescription associated with the space-time under consideration, we evaluate the non-zero components

[1] A vector field ξ which satisfied $\pounds_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$ is called Killing vector field.

of $H_{\mu}^{\nu\alpha}$

$$\begin{aligned}
H_{0}^{01} &= 2R', \\
H_{1}^{01} &= 2\dot{R}, \\
H_{2}^{02} &= f\left(\frac{f\dot{R}-R\dot{f}}{f^2}+\frac{R}{f}(2\dot{k}+\dot{\Omega})+rf\omega\dot{\omega}\right), \\
H_{3}^{02} &= -Rf^2\dot{\omega}, \\
H_{3}^{03} &= f\omega\left(-Rf\dot{\omega}+\frac{R}{f\omega}(2k+\dot{\Omega})+\frac{R\dot{f}}{\omega f^2}+\frac{\dot{R}}{f\omega}\right).
\end{aligned} \tag{3.3}$$

Using these components in (3.1), we get the energy and momentum densities as following

$$\theta_{0}^{0} = \frac{1}{8\pi}R'',$$

$$\theta_{1}^{0} = \frac{1}{8\pi}\dot{R}',$$

$$\theta_{2}^{0} = \theta_{3}^{0} = 0.$$

3.2 The Energy-Momentum Complex of Bergmann–Thomson

The Bergmann–Thomson energy-momentum complex [5] is given by

$$B^{\mu\nu} = \frac{1}{16\pi}\left[g^{\mu\alpha}\mathcal{B}_{\alpha}^{\nu\beta}\right]_{,\beta}, \tag{3.4}$$

where

$$\mathcal{B}_{\alpha}^{\nu\beta} = \frac{g_{\alpha\rho}}{\sqrt{-g}}[-g(g^{\nu\rho}g^{\beta\sigma}-g^{\beta\rho}g^{\nu\sigma})]_{,\sigma}.$$

B^{00} and B^{0i} $(i=1,2,3)$, are the energy and momentum density components. In order to calculate B^{00} and B^{0i} for Weyl metric, using Bergmann–Thomson energy-momentum complex, we require the following non-vanishing components of $\mathcal{B}_{\alpha}^{\nu\beta}$

$$\begin{aligned}
\mathcal{B}_{0}^{01} &= 2R', \\
\mathcal{B}_{1}^{01} &= 2\dot{R}, \\
\mathcal{B}_{2}^{02} &= f\left(\frac{f\dot{R}-R\dot{f}}{f^2}+\frac{R}{f}(2\dot{k}+\dot{\Omega})+rf\omega\dot{\omega}\right), \\
\mathcal{B}_{3}^{02} &= -Rf^2\dot{\omega}, \\
\mathcal{B}_{3}^{03} &= f\omega\left(-Rf\dot{\omega}+\frac{R}{f\omega}(2k+\dot{\Omega})+\frac{R\dot{f}}{\omega f^2}+\frac{\dot{R}}{f\omega}\right).
\end{aligned} \tag{3.5}$$

Using the components (3.5) and (2.4) in (3.4), we get the energy and momentum densities for the space-time under consideration, respectively, as follows

$$\begin{aligned}
B^{00} &= \frac{e^{-(2k+\Omega)}}{8\pi}\left[(2k'+\Omega')R'-R''\right], \\
B^{01} &= \frac{e^{-(2k+\Omega)}}{8\pi}\left[\dot{R}'-(2\dot{k}+\dot{\Omega})R'\right], \\
B^{02} &= B^{03} = 0.
\end{aligned} \tag{3.6}$$

3.3 Landau–Lifshitz's Energy-Momentum Complex

The energy-momentum complex of Landau–Lifshitz [3] is

$$L^{\mu\nu} = \frac{1}{16\pi} \S^{\mu\alpha\nu\beta}{}_{,\alpha\beta}, \tag{3.7}$$

where $\S^{\mu\alpha\nu\beta}$ with symmetries of the Riemann tensor and is defined by

$$\S^{\mu\alpha\nu\beta} = -g(g^{\mu\nu} g^{\alpha\beta} - g^{\mu\beta} g^{\alpha\nu}). \tag{3.8}$$

The quantity L^{00} represents the energy density of the whole physical system including gravitation and L^{0i} $(i = 1, 2, 3)$, represents the components of the total momentum (energy current) density.

In order to evaluate the energy and momentum densities in Landau–Lifshitz's prescription associated with the metric (2.2), we evaluate the non-zero components of $\S^{\mu\alpha\nu\beta}$

$$\begin{aligned} \S^{0101} &= -R^2, \\ \S^{0202} &= -\frac{R}{f}(f^2\omega^2 + 1)e^{2k+\Omega}, \\ \S^{0203} &= f\omega R e^{2k+\Omega}, \\ \S^{0303} &= -Rfe^{2k+\Omega}. \end{aligned} \tag{3.9}$$

Using these components in equation (3.7), we get the energy and momentum densities as following

$$\begin{aligned} L^{00} &= -\frac{1}{8\pi}(RR'' + R'^2), \\ L^{01} &= \frac{1}{8\pi}[R\dot{R}' + \dot{R}R'], \\ L^{02} &= L^{03} = 0. \end{aligned} \tag{3.10}$$

4 Energy-Momentum Complexes in Teleparallel Gravity

The name of teleparallel gravity is normally used to denote the general three-parameter theory introduced in [39]. The teleparallel gravity equivalent of general relativity [40] can indeed be understand as a gauge theory for the translation group based on Weitzenböck geometry [41]. In this theory, the gravitational interaction is described by a force similar to Lorentz force equation of electrodynamics, with torsion playing the role of force [42] and the curvature tensor vanishes identically.

Let us start by reviewing the fundamentals of the teleparallel equivalent of general relativity (see for example [39, 42, 43]). A gauge transformations is defined as a local translation of the tangent coordinates,

$$x^a \to x'^a = x^a + \eta^a,$$

where $\eta^a(x^\nu)$ are the transformation parameter. For an infinitesimal transformation, we have

$$\delta x^a = \delta\eta^c P_c x^a$$

with $P_c = \frac{\partial}{\partial x}$ the generators of transformation. The gauge covariant derivative of a general matter field $\Phi(x^\nu)$ is

$$D_\nu \Phi = h^a_{\ \nu} \partial_a \Phi,$$

where

$$h^a_{\ \nu} = \partial_\nu x^a + B^a_{\ \nu} \tag{4.1}$$

is a non-trivial tetrad field, $B^a_{\ \nu}$ is the translation gauge potential.

The relation (4.1) satisfies the orthogonality condition

$$h^a_{\ \nu} h_a^{\ \nu} = \delta^\nu_\mu. \tag{4.2}$$

Notice that, where as the tangent space indices are raised and lowered with the Minkowski metric η_{ab}, the space-time indices are raised and lowered with the space-time metric

$$g_{\mu\nu} = \eta_{ab} h^a_{\ \mu} h^b_{\ \nu}. \tag{4.3}$$

A nontrivial tetrad field induces on space-time a teleparallel structure which is directly related to the presence of the gravitational field. The parallel transport of the tetrad $h^a_{\ \nu}$ between two neighboring points is encoded in the covariant derivative

$$\nabla_\mu h^a_{\ \nu} = \partial h^a_{\ \nu} - \Gamma^\alpha_{\mu\nu} h^a_{\ \alpha},$$

where

$$\Gamma^\alpha_{\mu\nu} = h_a^{\ \alpha} \partial_\mu h^a_{\ \nu} \tag{4.4}$$

is the Weitzenböck connection. This connection is presenting torsion, but no curvature. The torsion of the Weitzenböck connection is defined by

$$T^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu}. \tag{4.5}$$

The Lagrangian of the teleparallel equivalent of general relativity is given by

$$\pounds = \pounds_G + \pounds_M = \frac{c^4 h}{16\pi G} S^{\rho\mu\nu} T_{\rho\mu\nu} + \$_M,$$

where $h = \det(h^a_{\ \mu}, \pounds_M)$ is the Lagrangian of a source field and

$$S^{\rho\mu\nu} = c_1 T^{\rho\mu\nu} + \frac{c_2}{2}\left(T^{\mu\rho\nu} - T^{\nu\rho\mu}\right) + \frac{c_3}{2}\left(g^{\rho\nu} T^{\sigma\mu}_{\ \ \sigma} - g^{\mu\rho} T^{\sigma\nu}_{\ \ \sigma}\right) \tag{4.6}$$

is a tensor written in terms of Weitzenböck connection. In the above form c_1, c_2 and c_3 are the three dimensionalless coupling constants of teleparallel gravity.

For the so called teleparallel equivalent of general relativity, the specific choice of these constants are given by [39]

$$c_1 = \frac{1}{4}, \qquad c_2 = \frac{1}{2}, \qquad c_3 = -1. \tag{4.7}$$

The energy-momentum complexes of Einstein, Bergmann–Thomson and Landau–Lifshitz in teleparallel gravity, respectively, are given by [32]

$$\begin{aligned} hE^{\mu}{}_{\nu} &= \frac{1}{4\pi}\partial_{\lambda}(\mho_{\nu}{}^{\mu\lambda}), \\ hB^{\mu\nu} &= \frac{1}{4\pi}\partial_{\lambda}(g^{\mu\beta}\mho_{\beta}{}^{\nu\lambda}), \\ hL^{\mu\nu} &= \frac{1}{4\pi}\partial_{\lambda}(hg^{\mu\beta}\mho_{\beta}{}^{\nu\lambda}), \end{aligned} \tag{4.8}$$

where $\mho_{\nu}{}^{\mu\lambda}$ is the Freud's super-potential and defined as follows

$$\mho_{\nu}{}^{\mu\lambda} = hS_{\nu}{}^{\mu\lambda}. \tag{4.9}$$

The energy and momentum distributions in the above complexes, respectively, are

$$\begin{aligned} P_{\mu}^{E} &= \int_{\Sigma} hE^{0}{}_{\mu} d^3x, \\ P_{\mu}^{BT} &= \int_{\Sigma} hB^{0}{}_{\mu} d^3x, \\ P_{\mu}^{LL} &= \int_{\Sigma} hL^{0}{}_{\mu} d^3x, \end{aligned} \tag{4.10}$$

where P_0 is the energy, P_i $(i = 1, 2, 3)$ are the momentum components and the integration hypersurface Σ is described by $x^0 = t$ constant.

5 Energy and Momentum Associated with the Metric (2.2) in Teleparallel Gravity

For the line element (2.2), using (4.3), we obtain the tetrad components

$$\begin{aligned} h^{0}{}_{0} &= e^{\frac{1}{2}(2k+\Omega)}, \\ h^{1}{}_{1} &= e^{\frac{1}{2}(2k+\Omega)}, \\ h^{2}{}_{2} &= \sqrt{Rf}, \\ h^{2}{}_{3} &= \omega\sqrt{Rf}, \\ h^{3}{}_{3} &= \sqrt{\frac{R}{f}}, \end{aligned} \tag{5.1}$$

its inverse

$$\begin{aligned} h_{0}{}^{0} &= e^{-\frac{1}{2}(2k+\Omega)}, \\ h_{1}{}^{1} &= e^{-\frac{1}{2}(2k+\Omega)}, \\ h_{2}{}^{2} &= d\frac{1}{\sqrt{Rf}}, \\ h_{3}{}^{2} &= -\omega\sqrt{\frac{R}{f}}, \\ h_{3}{}^{3} &= \sqrt{\frac{f}{R}}. \end{aligned} \tag{5.2}$$

$$h = \det(h^a{}_\mu) = Re^{(2k+\Omega)}. \tag{5.3}$$

Using the above tetrad and its inverse in (4.4), we get the following non-vanishing Weitzenböck connection components

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2}(2\dot{k} + \dot{\Omega}),\\
\Gamma^0_{01} &= \frac{1}{2}(2k' + \Omega'),\\
\Gamma^1_{10} &= \frac{1}{2}(2\dot{k} + \dot{\Omega}),\\
\Gamma^1_{11} &= \frac{1}{2}(2k' + \Omega'),\\
\Gamma^2_{20} &= \frac{R\dot{f} + f\dot{R}}{2Rf},\\
\Gamma^2_{21} &= \frac{Rf' + fR'}{2Rf},\\
\Gamma^2_{30} &= \frac{\omega\dot{f}}{f},\\
\Gamma^2_{31} &= \frac{\omega f'}{f},\\
\Gamma^3_{30} &= \frac{R\dot{f} - f\dot{R}}{2R^2},\\
\Gamma^3_{31} &= \frac{Rf' + fR'}{2R^2},
\end{aligned} \tag{5.4}$$

where dot and prime indicates derivative with respect to t and x, respectively. Using (4.5) and the above components, we find the following non-vanishing torsion components

$$\begin{aligned}
T^0_{10} &= -T^0_{01} = \frac{1}{2}(2k' + \Omega'),\\
T^1_{01} &= -T^1_{10} = \frac{1}{2}(2\dot{k} + \dot{\Omega}),\\
T^2_{02} &= -T^2_{20} = \frac{R\dot{f} + f\dot{R}}{2Rf},\\
T^2_{12} &= -T^2_{21} = \frac{Rf' + fR'}{2Rf},\\
T^2_{03} &= -T^2_{30} = \frac{\omega\dot{f}}{f},\\
T^2_{13} &= -T^2_{31} = \frac{\omega f'}{f},\\
T^3_{03} &= -T^3_{30} = \frac{f\dot{R} - R\dot{f}}{2Rf},\\
T^3_{13} &= -T^3_{31} = \frac{fR' - Rf'}{2Rf}.
\end{aligned} \tag{5.5}$$

Using these results into equation (4.6), the non-vanishing components of the tensor $S_\beta^{\ \mu\nu}$ are as following

$$
\begin{aligned}
S_0^{\ 01} &= \frac{R'}{2R} e^{-(2k+\Omega)}, \\
S_1^{\ 01} &= \frac{\dot{R}}{2R} e^{-(2k+\Omega)}, \\
S_2^{\ 02} &= e^{-(2k+\Omega)} \left[\frac{3\dot{f}}{f} - \frac{3}{4} f \dot{f} \omega^2 - \frac{\dot{R}}{2R} - \dot{k} - \frac{\dot{\Omega}}{2} \right], \\
S_3^{\ 02} &= -\frac{1}{4} e^{-(2k+\Omega)} \dot{f} \omega [\omega^2 + 1 + 2f^{-1}], \\
S_3^{\ 03} &= e^{-(2k+\Omega)} \left[\frac{3\dot{R}}{4R} + \frac{1}{4} f \dot{f} \omega^2 + \dot{k} + \frac{\dot{\omega}}{2} \right].
\end{aligned}
\tag{5.6}
$$

Using these components and the relation (4.9) in (4.8), we obtain the energy and momentum densities in the sense of Einstein, Bergmann–Thomson and Landau–Lifshitz, respectively, as follows

$$
\begin{aligned}
hE_0^0 &= \frac{1}{8\pi} R'', \\
hE_1^0 &= \frac{1}{8\pi} \dot{R}', \\
hE_2^0 &= hE_3^0 = 0.
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
hB^{00} &= \frac{e^{-(2k+\Omega)}}{8\pi} [(2k' + \Omega')R' - R''], \\
hB^{01} &= \frac{e^{-(2k+\Omega)}}{8\pi} [\dot{R}' - (2\dot{k} + \dot{\Omega})R'], \\
hB^{02} &= hB^{03} = 0.
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
hL^{00} &= -\frac{1}{8\pi} (RR'' + R'^2), \\
hL^{01} &= \frac{1}{8\pi} [R\dot{R}' + \dot{R}R'], \\
hL^{02} &= hL^{03} = 0.
\end{aligned}
\tag{5.9}
$$

These results agree with the results obtained in general relativity by using these different energy-momentum complexes.

6 Torsion Vector and Axial Torsion-Vector

The relation between the Weitzenböck connection $\Gamma^\sigma_{\mu\nu}$ and the Levi-Civita connection $\tilde{\Gamma}^\sigma_{\mu\nu}$ of the metric (4.3) is given by

$$\Gamma^\sigma_{\ \mu\nu} = \tilde{\Gamma}^\sigma_{\ \mu\nu} + K^\sigma_{\ \mu\nu},$$

where

$$\tilde{\Gamma}^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} [\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}]$$

and

$$K^{\sigma}_{\ \mu\nu} = \frac{1}{2}\big[T_{\mu\ \ \nu}^{\ \sigma} + T_{\nu\ \ \mu}^{\ \sigma} - T^{\sigma}_{\ \mu\nu}\big]$$

is the connection tensor with $T^{\sigma}_{\ \mu\nu}$ given by (4.5).

The torsion tensor can be decomposed into three irreducible parts under the group of global Lorentz transformations [39]. They are the tensor part

$$t_{\lambda\mu\nu} = \frac{1}{2}(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \frac{1}{6}(g_{\nu\lambda}V_{\mu} + g_{\mu\nu}V_{\lambda}) - \frac{1}{3}g_{\lambda\mu}V_{\nu},$$

the vector part

$$V_{\mu} = T^{\nu}_{\ \nu\mu}, \tag{6.1}$$

and the axial-vector part

$$A^{\mu} = h_a^{\ \mu}A^{a} = \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}T_{\nu\rho\sigma}. \tag{6.2}$$

Here the completely antisymmetric tensors $\varepsilon^{\mu\nu\rho\sigma}$ and $\varepsilon_{\mu\nu\rho\sigma}$ with respect to the coordinates basis are defined by [44]

$$\begin{aligned}\varepsilon^{\mu\nu\rho\sigma} &= \frac{1}{\sqrt{-g}}\delta^{\mu\nu\rho\sigma},\\ \varepsilon_{\mu\nu\rho\sigma} &= \sqrt{-g}\,\delta_{\mu\nu\rho\sigma},\end{aligned}$$

where $\delta^{\mu\nu\rho\sigma}$ and $\delta_{\mu\nu\rho\sigma}$ are the completely antisymmetric tensor densities of weight -1 and $+1$, respectively, with normalization $\delta^{0123} = +1$ and $\delta_{0123} = -1$.

Using (2.3) and (5.5) in equations (6.1) and (6.2), we get the non-vanishing components of the torsion vector in the following

$$\begin{aligned}V_0 &= -\frac{1}{2}\big(2\dot{k} + \dot{\Omega}\big) - \frac{\dot{R}}{R},\\ V_1 &= -\frac{1}{2}\big(2k' + \Omega'\big) - \frac{R'}{R},\end{aligned} \tag{6.3}$$

and axial-torsion vector

$$A^{\mu} = 0. \tag{6.4}$$

7 Discussion

The traditional manner in which physicists have identified a total density including the contribution from gravity has been that of re-costing the covariant conservation laws, $T^{\mu\nu}_{\ \ ;\nu}$, into the form of an ordinary vanishing divergence. Einstein himself changed the energy and momentum conservation law to

$$\frac{\partial}{\partial x^{\mu}}\big(\sqrt{-g}(T^{\mu}_{\nu} + t^{\mu}_{\nu})\big) = 0.$$

The pseudotensor, t^{μ}_{ν}, can be changed at will and hence there is an important ambiguity introduced regarding the value of energy and momentum densities. In fact, the question of the meaningfulness of energy localization in general relativity is raised. Misner et al. [45] argued that the energy is localizable only for spherical systems. Cooperstock and Sarracino [46] contradicted their viewpoint and argued that if the energy is localizable in spherical systems then it is also localizable for all systems. Bondi [47] expressed that a non-localizable form of energy is inadmissible in relativity and its location can in principle be found. In a series of papers, Cooperstock [48–51] hypothesized that in a curved space-time energy and momentum are confined to the region of non-vanishing energy-momentum tensor T_b^a and consequently the gravitational waves are not carriers of energy and momentum in vacuum space-times. This hypothesis has neither been proved nor disproved. There are many results support this hypothesis (see for example, [52–55]).

These difficulties are related to the lack of precise definition of a pseudotensor in the general relativity. As we mentioned in the introduction many physicists have introduced different definitions to solve this problem, but till now the problem stills unsolved.

One of the approaches to solve this problem, in the context of general relativity, is the quasilocal idea, which has been developed by Chang, Nester and Chen [56]. According to this idea, for each gravitational energy-momentum pseudotensor, there is a Hamiltonian boundary term. The energy-momentum defined by such a pseudotensor does not really depend on the local value of the reference frame, but only on the value of the reference frame on the boundary of a region, then its quasilocal character. This idea validates the pseudotensor approach to the gravitational energy-momentum problem.

The recent attempt to solve this problem is to replace the theory of general relativity by another theory, concentrated on the gauge theories for the translation group, the so called teleparallel equivalent of general relativity.

In this work we have explicitly evaluated the energy and momentum densities in an inhomogeneous cosmological solutions of the Einstein field equations. These solutions have an irrotational perfect fluid, which equation of state is $p = \rho$, as source. The energy and momentum densities are obtained in both the theory of general relativity and the theory of teleparallel gravity. We used different energy-momentum complexes, specifically these are the energy-momentum complexes of Einstein, Bergmann–Thomson and Landau–Lifshitz. We found first, in general relativity, that these definitions do not provide the same energy and momentum densities.

We were hoping that the theory of teleparallel gravity would solve this problem. Unfortunately, using the aforementioned energy-momentum complexes, in this theory these definitions do not provide also the same results for the energy and momentum densities.

In the context of the theory of teleparallel gravity, the components torsion vector and axial-vector torsion are obtained. For the space-time under consideration the axial-vector torsion vanishes identically.

References

1. Einstein, A.: Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **778** (1915)
2. Tolman, R.C.: Relativity, Thermodynamics and Cosmology, p. 227. Oxford University Press, Oxford (1934)
3. Landau, L.D., Lifshitz, E.M.: The Classical Theory of Fields, p. 317. Addison–Wesley, Reading (1951)
4. Papapetrou, A.: Proc. Roy. Ir. Acad. A **52**, 11 (1948)
5. Bergmann, P.G., Thomson, R.: Phys. Rev. **89**, 400 (1953)
6. Weinberg, S.: Gravitation and Cosmology: Principles and Applications of General Theory of Relativity, p. 165. Wiley, New York (1972)
7. Møller, C.: Ann. Phys. (N.Y.) **4**, 347 (1958)
8. Virbhadra, K.S.: Phys. Rev. D **41**, 1086 (1990)

9. Virbhadra, K.S.: Phys. Rev. D **42**, 1066 (1990)
10. Virbhadra, K.S.: Phys. Rev. D **42**, 2919 (1990)
11. Virbhadra, K.S.: Pramana J. Phys. **45**, 215 (1995)
12. Chamorro, A., Virbhadra, K.S.: Pramana J. Phys. **45**, 181 (1995)
13. Virbhadra, K.S., Parikh, J.C.: Phys. Lett. B **317**, 312 (1993)
14. Virbhadra, K.S., Parikh, J.C.: Phys. Lett. B **331**, 302 (1994)
15. Aguirregabiria, J.M., Chamorro, A., Virbhadra, K.S.: Gen. Relativ. Gravit. **28**, 1393 (1996)
16. Virbhadra, K.S.: Phys. Rev. D **60**, 104041 (1999)
17. Xulu, S.S.: Astrophys. Space Sci. **283**, 23 (2003)
18. Vagenas, E.C.: Int. J. Mod. Phys. A **18**, 5781 (2003)
19. Vagenas, E.C.: Int. J. Mod. Phys. A **18**, 5949 (2003)
20. Vagenas, E.C.: Mod. Phys. Lett. A **19**, 213 (2004)
21. Vagenas, E.C.: Int. J. Mod. Phys. D **14**, 573 (2005)
22. Vagenas, E.C.: Mod. Phys. Lett. A **21**, 1947 (2006)
23. Xulu, S.S.: Mod. Phys. Lett. A **15**, 1511 (2000)
24. Chamorro, A., Virbhadra, K.S.: Int. J. Mod. Phys. D **5**, 251 (1996)
25. Radinschi, I.: Mod. Phys. Lett. A **16**, 673 (2001)
26. Virbhadra, K.S.: Int. J. Mod. Phys. A **12**, 4831 (1997)
27. Gad, R.M.: Nuovo Cimento B **121**, 161 (2006)
28. Gad, R.M.: Astrophys. Space Sci. **293**, 453 (2004)
29. Gad, R.M.: Mod. Phys. Lett. A **19**, 1847 (2004)
30. Gad, R.M.: Astrophys. Space Sci. **302**, 141 (2006)
31. Gad, R.M.: Gen. Relativ. Gravit. **38**, 417 (2006)
32. Vargas, T.: Gen. Relativ. Gravit. **36**, 1255 (2004)
33. Salti, M.: Astrophys. Space Sci. **299**, 159 (2005)
34. Aydogdu, O., Salti, M.: Astrophys. Space Sci. **299**, 227 (2005)
35. Salti, M., Havare, A.: Int. J. Mod. Phys. A **20**, 2165 (2005)
36. Havare, A., Korunur, M., Salti, M.: Astrophys. Space Sci. **301**, 43 (2006)
37. Verdaguer, E.: Phys. Rep. **229**, 1 (1993)
38. Wainwright, J., Ince, W.C.W., Marshman, B.J.: Gen. Relativ. Gravit. **10**, 259 (1979)
39. Hayashi, K., Shirafuji, T.: Phys. Rev. D **19**, 3524 (1978)
40. Hammond, R.T.: Rep. Prog. Phys. **65**, 599 (2002)
41. Weitzenböck, R.: Invariantentheorie. Noordhoff, Gronningen (1923)
42. de Andrade, V.C., Pereira, J.G.: Phys. Rev. D **56**, 4689 (1997)
43. de Andrade, V.C., Guillen, L.C.T., Pereira, J.G.: Phys. Rev. Lett. **84**, 4533 (2000)
44. Møller, C.: The Theory of Relativity. Clarendon, Oxford (1952)
45. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation, p. 603. Freeman, New York (1973)
46. Cooperstock, F.I., Sarracino, R.S.: J. Phys. A **11**, 877 (1978)
47. Bondi, H.: Proc. Roy. Soc. Lond. A **427**, 249 (1990)
48. Cooperstock, F.I.: Found. Phys. **22**, 1011 (1992)
49. Cooperstock, F.I.: In: Mansouri, F., Scanio, J.J. (eds.) Topics in Quantum Gravity and Beyond: Pepers in Honor of L. Witten, p. 201. World Scientific, Singapore (1993)
50. Cooperstock, F.I.: In: Buitrago, et al. (eds.) Relativistic Astrophysics and Cosmology, p. 61. World Scientific, Singapore (1997)
51. Cooperstock, F.I.: Ann. Phys. **282**, 115 (2000)
52. Xulu, S.S.: Mod. Phys. Lett. A **15** 1511
53. Xulu, S.S.: Astrophys. Space Sci. **283**, 23 (2003)
54. Bringley, T.: Mod. Phys. Lett. A **17**, 175 (2002)
55. Gad, R.M.: Astrophys. Space Sci. **295**, 451 (2005)
56. Chang, C.C., Nester, J.M., Chen, C.M.: Phys. Rev. Lett. **83**, 1897 (1999)